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Approximating L^2 Invariants of Amenable Covering Spaces: A Combinatorial Approach

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In this paper, we prove that the L^2 Betti numbers of an amenable covering space can be approximated by the average Betti numbers of a regular exhaustion, proving a conjecture in [DM]. We also prove that an arbitrary amenable covering space of a finite simplicial complex is of determinant class. © 1998 Academic Press

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INTRODUCTION

Let Y be a connected simplicial complex and π be a discrete group of automorphisms of Y which acts freely and simplicially on Y so that $X = Y/\pi$ is a finite simplicial complex. Let $\mathscr{F}$ a finite subcomplex of Y, which is a fundamental domain for the action of π on Y. We assume that π is amenable. The Følner criterion for amenability of π enables one to get, cf. [Ad], a *regular exhaustion* $\{Y_m\}_{m=1}^{\infty}$, that is a sequence of finite subcomplexes of Y such that

(1) Y_m consists of N_m translates $g \cdot \mathscr{F}$ of $\mathscr{F}$ for $g \in \pi$.

(2) $Y = \bigcup_{m=1}^{\infty} Y_m$.

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(3) Let $\dot{N}_{m,\delta}$ denote the number of elements $g \in \pi$ which have distance (with respect to the word metric in π) less than or equal to δ from any element $g' \in \pi$ such that the intersection of $g' \cdot \mathscr{F}$ with the topological boundary ∂Y_m of Y_m is not empty. Then for every $\delta > 0$,

$$\lim_{m \to \infty} \frac{\dot{N}_{m,\delta}}{N_m} = 0.$$

One of our main results is

THEOREM 0.1 (Amenable Approximation Theorem). *Let Y be a connected simplicial complex. Suppose that π is amenable and that π acts freely and simplicially on Y so that $X = Y/\pi$ is a finite simplicial complex. Let $\{Y_m\}_{m=1}^{\infty}$ be a regular exhaustion of Y. Then*

$$\lim_{m \to \infty} \frac{b^j(Y_m)}{N_m} = b^j_{(2)}(Y : \pi) \qquad \textit{for all} \quad j \geqslant 0.$$

$$\lim_{m \to \infty} \frac{b^j(Y_m, \partial Y_m)}{N_m} = b^j_{(2)}(Y : \pi) \qquad \textit{for all} \quad j \geqslant 0.$$

Here $b^j(Y_m)$ denotes the jth Betti number of Y_m, $b^j(Y_m, \partial Y_m)$ denotes the jth Betti number of Y_m relative its boundary ∂Y_m and $b^j_{(2)}(Y : \pi)$ denotes the jth L^2 Betti number of Y. (See the next section for the definition of the L^2 Betti numbers of a manifold, and [W] for the notion of simplicial complex and associated cohomology).

This theorem proves the main conjecture in the introduction of an earlier paper [DM]. The combinatorial techniques used here replace the heat kernel approach used in [DM].

The paper of Eckmann [Ec] fore-shadowed our main result. Namely, Eckmann proves that if, for a fixed k, $\dim H^k(Y) < \infty$, then the ratios $b^k(Y_m)/N_m$ converge to zero. On the other hand, according to Cheeger and Gromov [CG], under the same assumption $b^k_{(2)} = 0$ as well. Thus one obtains a special case of Theorem 0.1, both sides being zero. In addition, Eckmann's paper contains another very suggestive result [Ec, Theorem 1.2] that the limit of the average Euler characteristic $\chi(Y_m)/N_m$ is equal to the Euler characteristic of the quotient X without any restrictions on the Betti numbers of Y. Januszkiewicz [J] and Roe [Roe] used an argument analogous to that of Eckmann in an analytic setting.

The equality of the limit of average Euler characteristic of the covering and the Euler characteristic of the base follow immediately from our

Theorem 0.1 and the L^2-index theorem. Eckmann's proof is elementary in that it makes no use of L^2-cohomology. However, motivated by his results on average Betti numbers and average Euler characteristic, he conjectured (private communication) that an approximation result like our Theorem 0.1 ought to be true. Thus Theorem 0.1 unifies and generalises some results of [Ec] and [CG] in a more general context.

Now let M be a smooth compact manifold, and X a triangulation of M. Let $\widetilde{M}$ be any normal covering space of M, and Y be the triangulation of $\widetilde{M}$ which projects down to X. There is a standing conjecture that any normal covering space of a finite simplicial complex is of *determinant class*, that is, the combinatorial Laplacian acting on L^2 cochains on Y has positive Fuglede–Kadison determinant (cf. Section 4 for the precise definition of determinant class and for a more detailed discussion of what follows). Then on $\widetilde{M}$, there are two notions of determinant class, one analytic and the other combinatorial. Using results of Efremov [E], Gromov and Shubin [GS], one observes as in [BFKM] that the combinatorial and analytic notions of determinant class coincide. It was proved in [BFK]) using estimates of Luck [Lu] that any *residually finite* normal covering space of a finite simplicial complex is of determinant class, which gave evidence supporting the conjecture. Our next main theorem says that any *amenable* normal covering space of a finite simplicial complex is of determinant class, which gives further evidence supporting the conjecture.

THEOREM 0.2 (Determinant Class Theorem). *Let Y be a connected simplicial complex. Suppose that π is amenable and that π acts freely and simplicially on Y so that $X = Y/\pi$ is a finite simplicial complex. Then Y is of determinant class.*

Our interest in this conjecture stems from work on L^2 torsion [CFM] (see also [BFKM]), in which L^2 torsion is a well defined element in the determinant line of the reduced L^2 cohomology, whenever the covering space is of determinant class. The results of this paper have also been used in [MR].

The paper is organized as follows. In the first section, some preliminaries on L^2 cohomology and amenable groups are presented. In Section 2, the main abstract approximation theorem is proved. We essentially use the combinatorial analogue of the principle of not feeling the boundary (cf. [DM]) in Lemma 2.3 and a finite dimensional result in [Lu], to prove this theorem. Section 3 contains the proof of the Amenable Approximation Theorem and some related approximation theorems. In Section 4, we prove that an arbitrary *amenable* normal covering space of a finite simplicial complex is of determinant class.

1. PRELIMINARIES

Our basic references for the preliminary material in this section are [Di] for von Neumann algebras and [At] and [Do] for L^2 cohomology.

Let π be a finitely generated discrete group and $\mathcal{U}(\pi)$ be the von Neumann algebra generated by the action of π on $\ell^2(\pi)$ via the left regular representation. It is the weak (or strong) closure of the complex group algebra of π, $\mathbb{C}(\pi)$ acting on $\ell^2(\pi)$ by left translation. The left regular representation is then a unitary representation $\rho: \pi \to \mathcal{U}(\pi)$. Let $\mathrm{Tr}_{\mathcal{U}(\pi)}$ be the faithful normal trace on $\mathcal{U}(\pi)$ defined by the inner product $\mathrm{Tr}_{\mathcal{U}(\pi)}(A) \equiv (A\,\delta_e, \delta_e)$ for $A \in \mathcal{U}(\pi)$ and where $\delta_e \in \ell^2(\pi)$ is given by $\delta_e(e) = 1$ and $\delta_e(g) = 0$ for $g \in \pi$ and $g \neq e$. Let $\mathcal{H}$ be a Hilbert space with a trivial action of π. The commutant theorem in [Di] states that the algebra of bounded operators on $\ell^2(\pi) \otimes \mathcal{H}$ which commute with the action of π is anti-isomorphic to $\mathcal{U}(\pi) \otimes B(\mathcal{H})$, where $B(\mathcal{H})$ denotes bounded operators on the Hilbert space $\mathcal{H}$. Then we can extend the trace $\mathrm{Tr}_{\mathcal{U}(\pi)}$ to a semifinite trace $\mathrm{Tr}_{\mathcal{U}(\pi)} \otimes \mathrm{Tr}$ on $\mathcal{U}(\pi) \otimes B(\mathcal{H})$, where Tr denotes the usual trace on $B(\mathcal{H})$. To lighten the notation, we will denote by $\mathrm{Tr}_{\mathcal{U}(\pi)}$ the trace $\mathrm{Tr}_{\mathcal{U}(\pi)} \otimes \mathrm{Tr}$. This trace is referred to as the *von Neumann trace*. Let $V \subseteq \ell^2(\pi) \otimes \mathcal{H}$ be a closed π invariant subspace. Let P_V denote the orthogonal projection onto V. Then we recall that the *von Neumann dimension* of V is defined as $\dim_\pi(V) = \mathrm{Tr}_{\mathcal{U}(\pi)}(P_V)$. It can be shown that $\dim_\pi(V)$ is independent of the choice of the π-invariant embedding of V as a subspace of $\ell^2(\pi) \otimes \mathcal{H}$, where $\mathcal{H}$ is as above.

Let Y be a simplicial complex, and $|Y|_j$ denote the set of all j-simplices in Y. Regarding the orientation of simplices, we use the following convention. For each j-simplex $\sigma \in |Y|_j$, we identify σ with any other j-simplex which is obtained by an *even* permutation of the vertices in σ, whereas we identify $-\sigma$ with any other j-simplex which is obtained by an *odd* permutation of the vertices in σ. Suppose that π acts freely and simplicially on Y so that $X = Y/\pi$ is a finite simplicial complex. Let $\mathcal{F}$ a finite subcomplex of Y, which is a fundamental domain for the action of π on Y. Let $C^j(Y)$ denote the space of finitely supported j-cochains on Y (cf. [W]), where one thinks of $f \in C^j(Y)$ as a function on the j-simplices of Y, which vanishes except on a finite number of j-simplices. There is a natural inner product $\langle\ ,\ \rangle$ on $C^j(Y)$ which is defined as follows:

$$\langle f, g\rangle = \sum_{\sigma \in |Y|_j} f(\sigma)\,\overline{g(\sigma)} \qquad \text{for all} \quad f, g \in C^j(Y).$$

The Hilbert space of square summable cochains on Y, denoted by $C^j_{(2)}(Y)$, is by definition the completion of $C^j(Y)$ with respect to the norm defined by the inner product $\langle\ ,\ \rangle$. Since π acts freely on Y, one can see that there is an isomorphism of Hilbert $\ell^2(\pi)$ modules,

$$C^j_{(2)}(Y) \cong C^j(X) \otimes \ell^2(\pi),$$

which depends on a choice of fundamental domain for the action of π on Y. Here π acts trivially on $C^j(X)$ and via the left regular representation on $\ell^2(\pi)$. By a previous discussion, we see that there is a von Neumann trace for the bounded operators on $C^j_{(2)}(Y)$ which commute with π. Let

$$d_j\colon C^j_{(2)}(Y) \to C^{j+1}_{(2)}(Y)$$

denote the coboundary operator. It is a bounded linear operator which commutes with π, since we have assumed that X is a finite simplicial complex. Then the (reduced) L^2 cohomology groups of Y are defined as

$$H^j_{(2)}(Y) = \frac{\ker(d_j)}{\mathrm{cl}(\mathrm{im}(d_{j-1}))}$$

where $\mathrm{cl}(\mathrm{im}(d_{j-1}))$ denotes the closure of the image of d_{j-1}. Let d_j^* denote the Hilbert space adjoint of d_j, which is a bounded linear operator that commutes with π. One defines the combinatorial Laplacian $\Delta_j\colon C^j_{(2)}(Y) \to C^j_{(2)}(Y)$ as $\Delta_j = d_{j-1}(d_{j-1})^* + (d_j)^* d_j$. Therefore it is a bounded linear operator which commutes with π.

By the Hodge decomposition theorem in this context, there is an isomorphism of Hilbert $\ell^2(\pi)$ modules,

$$H^j_{(2)}(Y) \cong \ker(\Delta_j).$$

Let $P_j\colon C^j_{(2)}(Y) \to \ker \Delta_j$ denote the orthogonal projection to the kernel of the Laplacian. Then it is a bounded linear operator which commutes with π. The L^2 Betti numbers $b^j_{(2)}(Y\colon \pi)$ are defined as

$$b^j_{(2)}(Y\colon \pi) = \dim_\pi(\ker \Delta_j) = \mathrm{Tr}_{\mathscr{U}(\pi)}(P_j) < \infty.$$

These have been shown to the homotopy invariants of the pair (Y, π) (cf. [Do]).

Let Y_m be a finite simplicial subcomplex of the simplicial complex Y, $C^j(Y_m)$ the finite dimensional space of j-cochains on Y_m and $C^j(Y_m, \partial Y_m)$ the finite dimensional space of j-cochains of the relative simplicial complex $(Y_m, \partial Y_m)$, where ∂Y_m denotes the boundary of Y_m (cf. [W]). In addition, let $\Delta_j^{(m)}$ denote the combinatorial Laplacian on the space $C^j(Y_m)$ or $C^j(Y_m, \partial Y_m)$. We do use the same notation for the two Laplacians since all proofs work equally well for either case. Let $D_j(\sigma, \tau) = \langle \Delta_j \delta_\sigma, \delta_\tau \rangle$ denote the matrix coefficients of the Laplacian Δ_j and

$D_j^{(m)}(\sigma, \tau) = \langle \Delta_j^{(m)} \delta_\sigma, \delta_\tau \rangle$ denote the matrix coefficients of the Laplacian $\Delta_j^{(m)}$. It follows from the definition of the Laplacian that

$$(\Delta_j \sigma, \tau) = (d\sigma, d\tau) + (d^*\sigma, d^*\tau).$$

One sees easily that $(d\sigma, d\tau) = 0$ unless σ and τ have a codimension one face in common. Similarly, $(d^*\sigma, d^*\tau) \neq 0$ implies that there exists a simplex of dimension $j+1$ having both σ and τ among its faces. Note that in the latter case σ and τ have a common face of dimension $j-1$. We shall call two j-simplexes *adjacent* if they are not equal and have a codimension one face in common. The discussion above shows that $(\Delta\sigma, \tau) = 0$ if the two simplexes are not equal and not adjacent. This motivates the following definition (cf. [Elek]) of simplicial distance $d(\sigma, \tau)$ between two simplexes of equal dimension. The distance $d(\sigma, \tau)$ is equal to zero if and only if $\sigma = \tau$; $d(\sigma, \tau) = 1$ if and only if σ and τ are adjacent. Furthermore, the distance between σ and τ is equal to $k \geqslant 2$ if there exists a finite sequence $\sigma = \sigma_0, \sigma_1, ..., \sigma_k = \tau$, $d(\sigma_i, \sigma_{i+1}) = 1$ for $i = 0, ..., k-1$, and k is the minimal length of such a sequence.

Then one has the following, which is an easy generalization of Lemma 2.5 in [Elek] and follows immediately from the definition of combinatorial Laplacians and finiteness of the complex $X = Y/\pi$.

Lemma 1.1. *$D_j(\sigma, \tau) = 0$ whenever $d(\sigma, \tau) > 1$ and $D_j^{(m)}(\sigma, \tau) = 0$ whenever $d_m(\sigma, \tau) > 1$. There is also a positive constant C independent of σ, τ such that $D_j(\sigma, \tau) \leqslant C$ and $D_j^{(m)}(\sigma, \tau) \leqslant C$.*

Let $D_j^k(\sigma, \tau) = \langle \Delta_j^k \delta_\sigma, \delta_\tau \rangle$ denote the matrix coefficient of the kth power of the Laplacian, Δ_j^k, and $D_j^{(m)k}(\sigma, \tau) = \langle (\Delta_j^{(m)})^k \delta_\sigma, \delta_\tau \rangle$ denote the matrix coefficient of the kth power of the Laplacian, $\Delta_j^{(m)k}$. Then

$$D_j^k(\sigma, \tau) = \sum_{\sigma_1, ..., \sigma_{k-1} \in |Y|_j} D_j(\sigma, \sigma_1) \cdots D_j(\sigma_{k-1}, \tau)$$

and

$$D_j^{(m)k}(\sigma, \tau) = \sum_{\sigma_1, ..., \sigma_{k-1} \in |Y_m|_j} D_j^{(m)}(\sigma, \sigma_1) \cdots D_j^{(m)}(\sigma_{k-1}, \tau).$$

Then the following lemma follows easily from Lemma 1.1.

Lemma 1.2. *Let k be a positive integer. Then $D_j^k(\sigma, \tau) = 0$ whenever $d(\sigma, \tau) > k$ and $D_j^{(m)k}(\sigma, \tau) = 0$ whenever $d_m(\sigma, \tau) > k$. There is also a positive constant C independent of σ, τ such that $D_j^k(\sigma, \tau) \leqslant C^k$ and $D_j^{(m)k}(\sigma, \tau) \leqslant C^k$.*

Since π commutes with the Laplacian Δ_j^k, it follows that

$$D_j^k(\gamma\sigma, \gamma\tau) = D_j^k(\sigma, \tau) \tag{1.1}$$

for all $\gamma \in \pi$ and for all $\sigma, \tau \in |Y|_j$. The *von Neumann trace* of Δ_j^k is by definition

$$\mathrm{Tr}_{\mathcal{U}(\pi)}(\Delta_j^k) = \sum_{\sigma \in |X|_j} D_j^k(\tilde{\sigma}, \tilde{\sigma}), \tag{1.2}$$

where $\tilde{\sigma}$ denotes an arbitrarily chosen lift σ to Y. The trace is well-defined in view of (1.1).

1.1. *Amenable Groups*

Let d_1 be the word metric on π. Recall the following characterization of amenability due to Følner, see also [Ad].

Definition 1.3. A discrete group π is said to be *amenable* if there is a sequence of finite subsets $\{\Lambda_k\}_{k=1}^{\infty}$ such that for any fixed $\delta > 0$

$$\lim_{k \to \infty} \frac{\sharp\{\partial_\delta \Lambda_k\}}{\sharp\{\Lambda_k\}} = 0$$

where $\partial_\delta \Lambda_k = \{\gamma \in \pi : d_1(\gamma, \Lambda_k) < \delta \text{ and } d_1(\gamma, \pi - \Lambda_k) < \delta\}$ is a δ-neighborhood of the boundary of Λ_k. Such a sequence $\{\Lambda_k\}_{k=1}^{\infty}$ is called a *regular sequence* in π. If in addition $\Lambda_k \subset \Lambda_{k+1}$ for all $k \geqslant 1$ and $\bigcup_{k=1}^{\infty} \Lambda_k = \pi$, then the sequence $\{\Lambda_k\}_{k=1}^{\infty}$ is called a *regular exhaustion* in π.

Examples of amenable groups are:

(1) finite groups;

(2) abelian groups;

(3) nilpotent groups and solvable groups;

(4) groups of subexponential growth;

(5) subgroups, quotient groups and extensions of amenable groups;

(6) the union of an increasing family of amenable groups.

Free groups and fundamental groups of closed negatively curved manifolds are *not* amenable.

Let π be a finitely generated amenable discrete group, and $\{\Lambda_m\}_{m=1}^{\infty}$ a regular exhaustion in π. Then it defines a regular exhaustion $\{Y_m\}_{m=1}^{\infty}$ of Y.

Let $\{P_j(\lambda) \colon \lambda \in [0, \infty)\}$ denote the right continuous family of spectral projections of the Laplacian Δ_j. Since Δ_j is π-equivariant, so are $P_j(\lambda) = \chi_{[0,\lambda]}(\Delta_j)$, for $\lambda \in [0, \infty)$. Let $F \colon [0, \infty) \to [0, \infty)$ denote the spectral density function,

$$F(\lambda) = \mathrm{Tr}_{\mathscr{U}(\pi)}(P_j(\lambda)).$$

Observe that the jth L^2 Betti number of Y is also given by

$$b^j_{(2)}(Y \colon \pi) = F(0).$$

We have the spectral density function for every dimension j but we do not indicate explicitly this dependence. All our arguments are performed with a fixed value of j.

Let $E_m(\lambda)$ denote the number of eigenvalues μ of $\Delta^{(m)}_j$ satisfying $\mu \leqslant \lambda$ and which are counted with multiplicity. We may sometimes omit the subscript j on $\Delta^{(m)}_j$ and Δ_j to simplify the notation.

We next make the following definitions,

$$F_m(\lambda) = \frac{E_m(\lambda)}{N_m}$$

$$\bar{F}(\lambda) = \limsup_{m \to \infty} F_m(\lambda)$$

$$\underline{F}(\lambda) = \liminf_{m \to \infty} F_m(\lambda)$$

$$\bar{F}^+(\lambda) = \lim_{\delta \to +0} \bar{F}(\lambda + \delta)$$

$$\underline{F}^+(\lambda) = \lim_{\delta \to +0} \underline{F}(\lambda + \delta).$$

2. MAIN TECHNICAL THEOREM

THEOREM 2.1. *Let π be countable, amenable group. In the notation of* 1.1, *Section* 1, *one has the following*:

(i) $F(\lambda) = \bar{F}^+(\lambda) = \underline{F}^+(\lambda)$;

(ii) *$\bar{F}$ and $\underline{F}$ are right continuous at zero and we have the equalities*

$$\begin{aligned}\bar{F}(0) = \bar{F}^+(0) = F(0) &= \underline{F}(0) = \underline{F}^+(0)\\ &= \lim_{m \to \infty} F_m(0) = \lim_{m \to \infty} \frac{\sharp\{E_m(0)\}}{N_m};\end{aligned}$$

(iii) *suppose that* $0<\lambda<1$, *then there is a constant* $K>1$ *such that*

$$F(\lambda)-F(0)\leqslant -a\,\frac{\log K^2}{\log \lambda}.$$

To prove this Theorem, we will first prove a number of prelimenary lemmas.

LEMMA 2.2. *There exists a positive number K such that the operator norms of* Δ_j *and of* $\Delta_j^{(m)}$ *for all* $m=1, 2, \ldots$ *are smaller than* K^2.

Proof. The proof is similar to that in [Lu], Lemma 2.5 and uses Lemma 1.1 together with uniform local finiteness of Y. More precisely we use the fact that the number of j-simplexes in Y at distance at most one from a j-simplex σ can be bounded *independently* of σ, say, $\sharp\{\tau\in |Y|_j : d(\tau, \sigma)\leqslant 1\}\leqslant b$. *A fortiori* the same is true (with the same constant a) for Y_m for all m. We now estimate the ℓ^2 norm of $\Delta\kappa$ for a cochain $\kappa=\sum_\sigma a_\sigma\sigma$ (having identified a simplex σ with the dual cochain). Now

$$\Delta\kappa=\sum_\sigma\left(\sum_\tau D(\sigma,\tau)\,a_\tau\right)\sigma$$

so that

$$\sum_\sigma\left(\sum_\tau D(\sigma,\tau)\,a_\tau\right)^2\leqslant\sum_\sigma\left(\sum_{d(\sigma,\tau)\leqslant 1} D(\sigma,\tau)^2\right)\left(\sum_{d(\sigma,\tau)\leqslant 1} a_\tau^2\right)$$

$$\leqslant C^2 b\sum_\sigma\sum_{d(\sigma,\tau)\leqslant 1} a_\tau^2,$$

where we have used Lemma 1.1 and Cauchy–Schwartz inequality. In the last sum above, for every simplex σ, a_σ^2 appears at most b times. This proves that $\|\Delta\kappa\|^2\leqslant C^2b^2\,\|\kappa\|^2$. Identical estimate holds (with the same proof) for $\Delta^{(m)}$ which yields the lemma if we set $K=\sqrt{Cb}$. ∎

Observe that Δ_j can be regarded as a matrix with entries in $\mathbb{Z}[\pi]$, since by definition, the coboundary operator d_j is a matrix with entries in $\mathbb{Z}[\pi]$, and so is its adjoint d_j^* as it is equal to the simplicial boundary operator. There is a natural trace for matrices with entries in $\mathbb{Z}[\pi]$, viz.

$$\mathrm{Tr}_{\mathbb{Z}[\pi]}(A)=\sum_i \mathrm{Tr}_{\mathcal{U}(\pi)}(A_{i,i}).$$

The following lemma can be regarded as a combinatorial analogue of the principle of not feeling the boundary, cf. [DM].

LEMMA 2.3. *Let π be an amenable group and let $p(\lambda)=\sum_{r=0}^{d} a_r \lambda^r$ be a polynomial. Then,*

$$\operatorname{Tr}_{\mathbb{Z}[\pi]}(p(\Delta_j)) = \lim_{m\to\infty} \frac{1}{N_m}\, Tr_{\mathbb{C}}\left(p\left(\Delta_j^{(m)}\right)\right).$$

Proof. First observe that if $\sigma \in |Y_m|_j$ is such that $d(\sigma, \partial Y_m) > k$, then Lemma 1.2 implies that

$$D_j^k(\sigma,\sigma) = \langle \Delta_j^k\, \delta_\sigma, \delta_\sigma\rangle = \langle \Delta_j^{(m)k}\, \delta_\sigma, \delta_\sigma\rangle = D_j^{(m)k}(\sigma,\sigma).$$

By (1.1) and (1.2)

$$\operatorname{Tr}_{\mathbb{Z}[\pi]}(p(\Delta_j)) = \frac{1}{N_m} \sum_{\sigma\in |Y_m|_j} \langle p(\Delta_j)\, \sigma, \sigma\rangle.$$

Therefore we see that

$$\left|\operatorname{Tr}_{\mathbb{Z}[\pi]}(p(\Delta_j)) - \frac{1}{N_m}\operatorname{Tr}_{\mathbb{C}}\left(p\left(\Delta_j^{(m)}\right)\right)\right|$$
$$\leqslant \frac{1}{N_m} \sum_{r=0}^{d} |a_r| \sum_{\substack{\sigma\in |Y_m|_j \\ d(\sigma,\partial Y_m)\leqslant d}} (D^r(\sigma,\sigma) + D^{(m)r}(\sigma,\sigma)).$$

Observe that the finiteness of $X = Y/\pi$ implies that there exists a positive integer $\delta = \delta(d)$ and a positive constant α such that the number of simplexes in the sum above is at most $\alpha \dot{N}_{m,d}$. Using Lemma 1.2, we see that there is a positive constant C such that

$$\left|\operatorname{Tr}_{\mathbb{Z}[\pi]}(p(\Delta_j)) - \frac{1}{N_m}\operatorname{Tr}_{\mathbb{C}}(p(\Delta_j^{(m)}))\right| \leqslant 2\,\frac{\alpha \dot{N}_{m,\delta}}{N_m} \sum_{r=0}^{d} |a_r|\, C^r.$$

The proof of the lemma is completed by taking the limit as $m\to\infty$. ∎

We next recall the following abstract lemma of Lück [Lu].

LEMMA 2.4. *Let $p_n(\mu)$ be a sequence of polynomials such that for the characteristic function of the interval $[0,\lambda]$, $\chi_{[0,\lambda]}(\mu)$, and an appropriate real number L,*

$$\lim_{n\to\infty} p_n(\mu) = \chi_{[0,\lambda]}(\mu) \qquad \text{and} \qquad |p_n(\mu)| \leqslant L$$

holds for each $\mu \in [0, \|\Delta_j\|^2]$. *Then*

$$\lim_{n\to\infty} \operatorname{Tr}_{\mathbb{Z}[\pi]}(p_n(\Delta_j)) = F(\lambda).$$

Lemmas 2.3 and 2.4 give

$$F(\lambda) = \lim_{n\to\infty} \operatorname{Tr}_{\mathbb{Z}[\pi]}(p_n(\Delta_j)) = \lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{N_m} \operatorname{Tr}_{\mathbb{C}}\left(p_n\left(\Delta_j^{(m)}\right)\right).$$

If one could interchange the two limits on the right-hand side above, the proof of Theorem 2.1 would be complete. The following lemma of Lück [Lu] justifies interchange of the limits.

LEMMA 2.5. *Let* $G\colon V \to W$ *be a linear map of finite dimensional Hilbert spaces* V *and* W. *Let* $p(t) = \det(t - G^*G)$ *be the characteristic polynomial of* G^*G. *Then* $p(t)$ *can be written as* $p(t) = t^k q(t)$ *where* $q(t)$ *is a polynomial with* $q(0) \neq 0$. *Let* K *be a real number*, $K \geqslant \max\{1, \|G\|\}$ *and* $C > 0$ *be a positive constant with* $|q(0)| \geqslant C > 0$. *Let* $E(\lambda)$ *be the number of eigenvalues* μ *of* G^*G, *counted with multiplicity, satisfying* $\mu \leqslant \lambda$. *Then for* $0 < \lambda < 1$, *the following estimate is satisfied.*

$$\frac{\{E(\lambda)\} - \{E(0)\}}{\dim_{\mathbb{C}} V} \leqslant \frac{-\log C}{(-\log \lambda)\dim_{\mathbb{C}} V} + \frac{\log K^2}{-\log \lambda}.$$

Proof of Theorem 2.1. Fix $\lambda \geqslant 0$ and define for $n \geqslant 1$ a continuous function $f_n\colon \mathbb{R} \to \mathbb{R}$ by

$$\begin{cases} f_n(\mu) = 1 + \dfrac{1}{n} & \text{if} \quad \mu \leqslant \lambda \\[2ex] 1 + \dfrac{1}{n} - n(\mu - \lambda) & \text{if} \quad \lambda \leqslant \mu \leqslant \lambda + \dfrac{1}{n} \\[2ex] \dfrac{1}{n} & \text{if} \quad \lambda + \dfrac{1}{n} \leqslant \mu \end{cases}$$

Then clearly $\chi_{[0,\lambda]}(\mu) < f_{n+1}(\mu) < f_n(\mu)$ and $f_n(\mu) \to \chi_{[0,\lambda]}(\mu)$ as $n \to \infty$ for all $\mu \in [0, \infty)$. For each n, choose a polynomial p_n such that $\chi_{[0,\lambda]}(\mu) < p_n(\mu) < f_n(\mu)$ holds for all $\mu \in [0, K^2]$. We can always find such a polynomial by a sufficiently close approximation of f_{n+1}. Hence

$$\chi_{[0,\lambda]}(\mu) < p_n(\mu) < 2$$

and

$$\lim_{n \to \infty} p_n(\mu) = \chi_{[0,\lambda]}(\mu)$$

for all $\mu \in [0, K^2]$. Recall that $E_m(\lambda)$ denotes the number of eigenvalues μ of $\Delta_j^{(m)}$ satisfying $\mu \leqslant \lambda$ and counted with multiplicity. Note that $\|\Delta_j^{(m)}\| \leqslant K^2$ by Lemma 2.2.

$$\begin{aligned}
&\frac{1}{N_m} \operatorname{Tr}_{\mathbb{C}}(p_n(\Delta_j^{(m)})) \\
&= \frac{1}{N_m} \sum_{\mu \in [0, K^2]} p_n(\mu) \\
&= \frac{E_m(\lambda)}{N_m} + \frac{1}{N_m} \left\{ \sum_{\mu \in [0, \lambda]} (p_n(\mu) - 1) + \sum_{\mu \in (\lambda, \lambda + 1/n]} p_n(\mu) + \sum_{\mu \in (\lambda + 1/n, K^2]} p_n(\mu) \right\}
\end{aligned}$$

Hence, we see that

$$F_m(\lambda) = \frac{E_m(\lambda)}{N_m} \leqslant \frac{1}{N_m} \operatorname{Tr}_{\mathbb{C}}(p_n(\Delta_j^{(m)})). \tag{2.1}$$

In addition,

$$\begin{aligned}
&\frac{1}{N_m} \operatorname{Tr}_{\mathbb{C}}(p_n(\Delta_j^{(m)})) \\
&\qquad \leqslant \frac{E_m(\lambda)}{N_m} + \frac{1}{N_m} \sup\{p_n(\mu) - 1 : \mu \in [0, \lambda]\}\ E_m(\lambda) \\
&\qquad\quad + \frac{1}{N_m} \sup\{p_n(\mu) : \mu \in [\lambda, \lambda + 1/n]\} (E_m(\lambda + 1/n) - E_m(\lambda)) \\
&\qquad + \frac{1}{N_m} \sup\{p_n(\mu) : \mu \in [\lambda + 1/n, K^2]\} (E_m(K^2) - E_m(\lambda + 1/n)) \\
&\qquad \leqslant \frac{E_m(\lambda)}{N_m} + \frac{E_m(\lambda)}{nN_m} + \frac{(1 + 1/n)(E_m(\lambda + 1/n) - E_m(\lambda))}{N_m} \\
&\qquad\quad + \frac{(E_m(K^2) - E_m(\lambda + 1/n))}{nN_m} \\
&\qquad \leqslant \frac{E_m(\lambda + 1/n)}{N_m} + \frac{1}{n} \frac{E_m(K^2)}{N_m} \\
&\qquad \leqslant F_m(\lambda + 1/n) + \frac{a}{n}
\end{aligned}$$

since $E_m(K^2) = \dim C^j(Y_m) \leqslant aN_m$ for a positive constant a independent of m. It follows that

$$\frac{1}{N_m} \mathrm{Tr}_{\mathbb{C}}(p_n(\Delta_j^{(m)})) \leqslant F_m(\lambda + 1/n) + \frac{a}{n}. \tag{2.2}$$

Taking the limit inferior in (2.2) and the limit superior in (2.1), as $m \to \infty$, we get that

$$\bar{F}(\lambda) \leqslant \mathrm{Tr}_{\mathbb{Z}[\pi]}(p_n(\Delta_j)) \leqslant \underline{F}(\lambda + 1/n) + \frac{a}{n}. \tag{2.3}$$

Taking the limit as $n \to \infty$ in (2.3) and using Lemma 2.4, we see that

$$\bar{F}(\lambda) \leqslant F(\lambda) \leqslant \underline{F}^+(\lambda).$$

For all $\varepsilon > 0$ we have

$$F(\lambda) \leqslant \underline{F}^+(\lambda) \leqslant \underline{F}(\lambda + \varepsilon) \leqslant \bar{F}(\lambda + \varepsilon) \leqslant F(\lambda + \varepsilon).$$

Since F is right continuous, we see that

$$F(\lambda) = \bar{F}^+(\lambda) = \underline{F}^+(\lambda)$$

proving the first part of Theorem 2.1.

Next we apply Lemma 2.5 to $\Delta_j^{(m)}$. Let $p_m(t)$ denote the characteristic polynomial of $\Delta_j^{(m)}$ and $p_m(t) = t^{r_m} q_m(t)$ where $q_m(0) \neq 0$. The matrix describing $\Delta_j^{(m)}$ has integer entries. Hence p_m is a polynomial with integer coefficients and $|q_m(0)| \geqslant 1$. By Lemma 2.2 and Lemma 2.5 there are constants K and $C = 1$ independent of m, such that

$$\frac{F_m(\lambda) - F_m(0)}{a} \leqslant \frac{\log K^2}{-\log \lambda}.$$

That is,

$$F_m(\lambda) \leqslant F_m(0) - \frac{a \log K^2}{\log \lambda}, \tag{2.4}$$

Taking limit inferior in (2.4) as $m \to \infty$ yields

$$\underline{F}(\lambda) \leqslant \underline{F}(0) - \frac{a \log K^2}{\log \lambda},$$

Passing to the limit as $\lambda \to +0$, we get that

$$\underline{F}(0) = \underline{F}^+(0) \qquad \text{and} \qquad \bar{F}(0) = \bar{F}^+(0).$$

We have seen already that $\bar{F}^+(0) = F(0) = \underline{F}(0)$, which proves part (ii) of Theorem 2.1. Since $-a \log K^2/\log \lambda$ is right continuous in λ,

$$\bar{F}^+(\lambda) \leqslant F(0) = \frac{a \log K^2}{\log \lambda}.$$

Hence part (iii) of Theorem 2.1 is also proved. ∎

We will need the following lemma in the proof of Theorem 0.2 in the last section. We follow the proof of Lemma 3.3.1 in [Lu].

LEMMA 2.6.

$$\int_{0+}^{K^2} \left\{ \frac{F(\lambda) - F(0)}{\lambda} \right\} d\lambda \leqslant \liminf_{m \to \infty} \int_{0+}^{K^2} \left\{ \frac{F_m(\lambda) - F_m(0)}{\lambda} \right\} d\lambda$$

Proof. By Theorem 2.1, and the monotone convergence theorem, one has

$$\begin{aligned}
\int_{0+}^{K^2} \left\{ \frac{F(\lambda) - F(0)}{\lambda} \right\} d\lambda &= \int_{0+}^{K^2} \left\{ \frac{\underline{F}(\lambda) - \underline{F}(0)}{\lambda} \right\} d\lambda \\
&= \int_{0+}^{K^2} \liminf_{m \to \infty} \left\{ \frac{F_m(\lambda) - F_m(0)}{\lambda} \right\} d\lambda \\
&= \int_{0+}^{K^2} \lim_{m \to \infty} \left(\inf \left\{ \frac{F_n(\lambda) - F_n(0)}{\lambda} \,\middle|\, n \geqslant m \right\} \right) d\lambda \\
&= \lim_{m \to \infty} \int_{0+}^{K^2} \inf \left\{ \frac{F_n(\lambda) - F_n(0)}{\lambda} \,\middle|\, n \geqslant m \right\} d\lambda \\
&\leqslant \liminf_{m \to \infty} \int_{0+}^{K^2} \left\{ \frac{F_m(\lambda) - F_m(0)}{\lambda} \right\} d\lambda. \quad ∎
\end{aligned}$$

3. PROOFS OF THE MAIN THEOREMS

In this section, we will prove the Amenable Approximation Theorem (Theorem 0.1) of the introduction. We will also prove some related spectral results.

Proof of Theorem 0.1 (Amenable Approximation Theorem). Observe that

$$\frac{b^j(Y_m)}{N_m} = \frac{\dim_{\mathbb{C}}(\ker(\Delta_j^{(m)}))}{N_m}$$
$$= F_m(0).$$

Also observe that

$$b^j_{(2)}(Y:\pi) = \dim_\pi(\ker(\Delta_j))$$
$$= F_m(0).$$

Therefore Theorem 0.1 follows from Theorem 2.1 after taking the limit as $m \to \infty$. ∎

Suppose that M is a compact Riemannian manifold and $\Omega^j_{(2)}(\tilde{M})$ denote the Hilbert space of square integrable j-forms on a normal covering space $\tilde{M}$, with transformation group π. The Laplacian $\tilde{\Delta}_j\colon \Omega^j_{(2)}(\tilde{M}) \to \Omega^j_{(2)}(\tilde{M})$ is essentially self-adjoint and has a spectral decomposition $\{\tilde{P}_j(\lambda)\colon \lambda \in [0,\infty)\}$ where each $\tilde{P}_j(\lambda)$ has finite von Neumann trace. The associated von Neumann spectral density function, $\tilde{F}(\lambda)$ is defined as

$$\tilde{F}\colon [0,\infty) \to [0,\infty), \qquad \tilde{F}(\lambda) = \mathrm{Tr}_{\mathscr{U}(\pi)}(\tilde{P}_j(\lambda)).$$

Note that $\tilde{F}(0) = b^j_{(2)}(\tilde{M}:\pi)$ and that the spectrum of $\tilde{\Delta}_j$ has a gap at zero if and only if there is a $\lambda > 0$ such that

$$\tilde{F}(\lambda) = \tilde{F}(0).$$

Suppose that π is an amenable group. Fix a triangulation X on M. Then the normal cover $\tilde{M}$ has an induced triangulation Y. Let Y_m denote be a subcomplex of Y such that $\{Y_m\}_{m=1}^{\infty}$ is a regular exhaustion of Y. Let $\Delta_j^{(m)}\colon C^j(Y_m,\mathbb{C}) \to C^j(Y_m,\mathbb{C})$ denote the combinatorial Laplacian, and let $E_j^{(m)}(\lambda)$ denote the number of eigenvalues μ of $\Delta_j^{(m)}$ which are less than or equal to λ. Under the hypotheses above we prove the following.

Theorem 3.1 (Gap Criterion). *The spectrum of $\tilde{\Delta}_j$ has a gap at zero if and only if there is a $\lambda > 0$ such that*

$$\lim_{m\to\infty} \frac{E_j^{(m)}(\lambda) - E_j^{(m)}(0)}{N_m} = 0.$$

Proof. Let $\Delta_j: C^j_{(2)}(Y) \to C^j_{(2)}(Y)$ denote the combinatorial Laplacian acting on L^2 *j*-cochains on *Y*. Then by [GS], [E], the von Neumann spectral density function F of the combinatorial Laplacian $\tilde{\Delta}_j$ and the von Neumann spectral density function $\tilde{F}$ of the analytic Laplacian Δ_{j} are dilatationally equivalent, that is, there are constants $C > 0$ and $\varepsilon > 0$ independent of λ such that for all $\lambda \in (0, \varepsilon)$,

$$F(C^{-1}\lambda) \leqslant \tilde{F}(\lambda) \leqslant F(C\lambda). \tag{3.1}$$

Observe that $E^{(m)}_j(\lambda)/N_m = F_m(\lambda)$. Therefore the theorem also follows from Theorem 2.1. ∎

There is a standing conjecture that the Novikov–Shubin invariants of a closed manifold are positive (see [E, ES, and GS] for its definition). The next theorem gives evidence supporting this conjecture, at least in the case of amenable fundamental groups.

THEOREM 3.2 (Spectral density Estimate). *There are constants $C > 0$ and $\varepsilon > 0$ independent of λ, such that for all $\lambda \in (0, \varepsilon)$*

$$\tilde{F}(\lambda) - \tilde{F}(0) \leqslant \frac{C}{-\log(\lambda)}.$$

Proof. This follows from Theorem 2.1 and Theorem 3.1 since $\tilde{\Delta}_j$ has a gap at zero if and only if $\tilde{F}_j(\lambda) = \tilde{F}_j(0)$ for some $\lambda > 0$. ∎

4. ON THE DETERMINANT CLASS CONJECTURE

There is a standing conjecture that any normal covering space of a finite simplicial complex is of determinant class. Our interest in this conjecture stems from work on L^2 torsion [CFM] (see also [BFKM]). The L^2 torsion is a well defined element in the determinant line of the reduced L^2 cohomology, whenever the covering space is of determinant class. In this section, we use the results of Section 2 to prove that any *amenable* normal covering space of a finite simplicial complex is of determinant class.

Recall that a covering space Y of a finite simplicial complex X is said to be of *determinant class* if, for $0 \leqslant j \leqslant n$,

$$-\infty < \int_{0^+}^{p} \log \lambda \, dF(\lambda) = \log(\det'_\pi \Delta_j),$$

where $\det'_\pi \Delta_j$ denotes the modified Fuglede–Kadison determinant (cf. [FK]) of Δ_j, that is, the Fuglede–Kadison determinant of Δ_j restricted to the orthogonal complement of its kernel, p is any number satisfying $p > \|\Delta_j\|$, $F(\lambda)$ denotes the von Neumann spectral density function of the combinatorial Laplacian Δ_j as defined in 1.1, Section 1.

Suppose that M is a compact Riemannian manifold and $\Omega^j_{(2)}(\tilde{M})$ denote the Hilbert space of square integrable j-forms on a normal covering space $\tilde{M}$, with transformation group π. The Laplacian $\tilde{\Delta}_j : \Omega^j_{(2)}(\tilde{M}) \to \Omega^j_{(2)}(\tilde{M})$ is essentially self-adjoint and the associated von Neumann spectral density function, $\tilde{F}(\lambda)$ is defined as in Section 3. Note that $\tilde{F}(0) = b^j_{(2)}(\tilde{M} : \pi)$ Then $\tilde{M}$ is said to be of *analytic-determinant class*, if, for $0 \leqslant j \leqslant n$,

$$-\infty < \int_{0^+}^{1} \log \lambda \, d\tilde{F}(\lambda),$$

where $\tilde{F}(\lambda)$ denotes the von Neumann spectral density function of the analytic Laplacian $\tilde{\Delta}_j$ as above. By results of Gromov and Shubin [GS], the condition that $\tilde{M}$ is of analytic-determinant class is independent of the choice of Riemannian metric on M.

Fix a triangulation X on M. Then the normal cover $\tilde{M}$ has an induced triangulation Y. Then $\tilde{M}$ is said to be of *combinatorial-determinant class* if Y is of determinant class. Using results of Efremov [E], and [GS] one sees that the condition that $\tilde{M}$ is of combinatorial-determinant class is independent of the choice of triangulation on M.

Using again results of [E] and [GS], one observes as in [BFKM] that the combinatorial and analytic notions of determinant class coincide, that is $\tilde{M}$ is of combinatorial-determinant class if and only if $\tilde{M}$ is of analytic-determinant class. Using results of [GS], one observes as in [BFKM] that the determinant class condition is a homotopy invariant of the pair $(\tilde{M}, \pi)$. The appendix of [BFK] contains a proof that every residually finite covering of a compact manifold is of determinant class. Their proof is based on Lück's approximation of von Neumann spectral density functions [Lu]. Since an analogous approximation holds in our setting (cf. Section 2), we can apply the argument of [BFK] to prove Theorem 0.2.

Proof of Theorem 0.2 (Determinant Class Theorem). Recall that the *normalized* spectral density functions

$$F_m(\lambda) = \frac{1}{N_m} E_j^{(m)}(\lambda)$$

are right continuous. Observe that $F_m(\lambda)$ are step functions and denote by $det' \, \Delta_j^{(m)}$ the modified determinant of $\Delta_j^{(m)}$, i.e., the product of all *nonzero* eigenvalues of $\Delta_j^{(m)}$. Let $a_{m,j}$ be the smallest nonzero eigenvalue and $b_{m,j}$

the largest eigenvalue of $\Delta_j^{(m)}$. Then, for any a and b, such that $0 < a < a_{m,j}$ and $b > b_{m,j}$,

$$\frac{1}{N_m} \log \det{}' \Delta_j^{(m)} = \int_a^b \log \lambda \, dF_m(\lambda). \tag{4.1}$$

Integration by parts transforms the Stieltjes integral $\int_a^b \log \lambda \, dF_m(\lambda)$ as follows.

$$\int_a^b \log \lambda \, dF_m(\lambda) = (\log b)(F_m(b) - F_m(0)) - \int_a^b \frac{F_m(\lambda) - F_m(0)}{\lambda} \, d\lambda. \tag{4.2}$$

As before, $F(\lambda)$ denotes the spectral density function of the operator Δ_j for a fixed j. Recall that $F(\lambda)$ is continuous to the right in λ. As before, denote by $\det'_\pi \Delta_j$ the modified Fuglede–Kadison determinant (cf. [FK]) of Δ_j, that is, the Fuglede–Kadison determinant of Δ_j restricted to the orthogonal complement of its kernel. Then it is given by the following Lebesgue–Stieltjes integral,

$$\log \det{}'_\pi \Delta_j = \int_{0^+}^{K^2} \log \lambda \, dF(\lambda)$$

with K as in Lemma 2.2, i.e., $\|\Delta_j\| < K^2$, where $\|\Delta_j\|$ is the operator norm of Δ_j.

Integrating by parts, one obtains

$$\log \det{}'_\pi(\Delta_j) = \log K^2(F(K^2) - F(0)) \\ + \lim_{\varepsilon \to 0^+} \left\{ (-\log \varepsilon)(F(\varepsilon) - F(0)) - \int_\varepsilon^{K^2} \frac{F(\lambda) - F(0)}{\lambda} \, d\lambda \right\}. \tag{4.3}$$

Using the fact that $\liminf_{\varepsilon \to 0} + (-\log \varepsilon(F(\varepsilon) - F(0)) \geqslant 0$ (in fact, this limit exists and is zero) and $F(\lambda) - F(0)/\lambda \geqslant 0$ for $\lambda > 0$, one sees that

$$\log \det{}'_\pi(\Delta_j) \geqslant (\log K^2)(F(k^2) - F(0)) - \int_{0^+}^{K^2} \frac{F(\lambda) - F(0)}{\lambda} \, d\lambda. \tag{4.4}$$

We now complete the proof of Theorem 0.2. The main ingredient is the estimate of $\log \det'_\pi(\Delta_j)$ in terms of $\log \det' \Delta_j^{(m)}$ combined with the fact that $\log \det' \Delta_j^{(m)} \geqslant 0$ as the determinant $\det' \Delta_j^{(m)}$ is a positive integer. By Lemma 2.2, there exists a positive number K, $1 \leqslant K < \infty$, such that, for $m \geqslant 1$,

$$\|\Delta_j^{(m)}\| \leqslant K^2 \qquad \text{and} \qquad \|\Delta_j\| \leqslant K^2.$$

By Lemma 2.6,

$$\int_{0^+}^{K_2} \frac{F(\lambda)-F(0)}{\lambda}\, d\lambda \leqslant \liminf_{m\to\infty} \int_{0^+}^{K^2} \frac{F_m(\lambda)-F_m(0)}{\lambda}\, d\lambda. \tag{4.5}$$

Combining (4.1) and (4.2) with the inequalities $\log \det' \Delta_j^{(m)} \geqslant 0$, we obtain

$$\int_{0^+}^{K^2} \frac{F_m(\lambda)-F_m(0)}{\lambda}\, d\lambda \leqslant (\log K^2)(F_m(K^2)-F_m(0)). \tag{4.6}$$

From (4.4), (4.5) and (4.6), we conclude that

$$\log \det'_\pi \Delta_j \geqslant (\log K^2)(F(K^2)-F(0)) - \liminf_{m\to\infty} (\log K^2)(F_m(K^2)-F_m(0)). \tag{4.7}$$

Now Theorem 2.1 yields

$$F(\lambda) = \lim_{\varepsilon\to 0^+} \liminf_{m\to\infty} F_m(\lambda+\varepsilon)$$

and

$$F(0) = \lim_{m\to\infty} F_m(0).$$

The last two equalities combined with (4.7) imply that $\log \det'_\pi \Delta_j \geqslant 0$. Since this is true for all $j=0, 1, ..., \dim Y$, Y is of determinant class. ∎

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REFERENCES

[Ad] T. Adachi, A note on the Følner condition of amenability, *Nagoya Math. J.* **131** (1993), 67–74.

[At] M. F. Atiyah, Elliptic operators, discrete groups and Von Neumann algebras, *Asterisque* **32–33** (1976), 43–72.

[BFK] D. Burghelea, L. Friedlander, and T. Kappeler, Torsion for manifolds with boundary and gluing formulas, IHES preprint, January, 1966.

[BFKM] D. Burghelea, L. Friedlander, T. Kappeler, and P. McDonald, Analytic and Reidemeister torsion for representations in finite type Hilbert modules, *GAFA* **6** (1996), 751–859.

[CFM] A. L. Carey, M. Farber, and V. Mathai, Determinant Lines, von Neumann algebras and L^2 torsion, *Crelle Journal* **484** (1997), 153–181.

[CG] J. Cheeger and M. Gromov, L^2 cohomology and group cohomology, *Topology* **25** (1986), 189–215.

[Di] J. Dixmier, "Von Neumann Algebras," Vol. 27, North-Holland, Amsterdam, 1981.

[Do] J. Dodziuk, De Rham–Hodge theory for L^2 cohomology of infinite coverings, *Topology* **16** (1977), 157–165.

[DM] J. Dodziuk and V. Mathai, Approximating L^2 invariants of amenable covering spaces: A heat kernel approach, *in* "Lipa's Legacy," *Contemp. Math.* **211** (1997), 151–167.

[Ec] B. Eckmann, Amenable groups and Euler characteristic, *Comment. Math. Helv.* **67** (1992), 383–393.

[Ec] A. V. Efremov, Cell decompositions and the Novikov-Shubin invariants, *Russian Math. Surveys* **46** (1991), 219–220.

[ES] D. V. Efremov and M. A. Shubin, Spectrum distribution function and variational principle for automorphic operators on hyperbolic space, Ecole Polytechnique Palaiseau, Centre de Mathematiques, 1989.

[Elek] G. Elek, Combinatorial heat kernels and index theorems, *J. Funct. Anal.* **129** (1995), 64–79.

[FK] B. Fuglede and R. V. Kadison, Determinant theory in finite factors, *Ann. Math.* **55** (1952), 520–530.

[GS] M. Gromov and M. A. Shubin, Von Neumann spectra near zero, *Geom. Funct. Anal.* **1** (1991), 375–404.

[J] T. Januszkiewicz, Characteristic invariants of non-compact manifolds, *Topology* **23** (1984), 289–302.

[Lu] W. Lück, Approximating L^2 invariants by their finite dimensional analogues, *Geom. Funct. Anal.* **4** (1994), 455–481.

[MR] V. Mathai and M. Rothenberg, On the homotopy invariance of L^2 torsion for covering spaces, *Proc. Amer. Math. Soc.* **126** (1998), 887–897.

[Roe] J. Roe, "Elliptic Operators, Topology and Asymptotic Methods," Pitman Research Notes Math. Series, Vol. 179, Longman, Harlow/New York, 1988.

[W] H. Whitney, "Geometric Integration Theory," Princeton Univ. Press, Princeton, NJ, 1957.